Journal of Algebra 335 (2011) 36–48

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Cyclic 2-structures and spaces of orderings of power series fields in two variables

Salma Kuhlmann [a,*], Murray Marshall [b], Katarzyna Osiak [c]

[a] Fachbereich Mathematik und Statistik, Universität Konstanz, D-78457 Konstanz, Germany
[b] Department of Mathematics and Statistics, University of Saskatchewan, Saskatoon, SK S7N 5E6, Canada
[c] Institute of Mathematics, Silesian University, Bankowa 14, 40-007 Katowice, Poland

ARTICLE INFO

Article history:
Received 13 December 2009
Available online 14 April 2011
Communicated by Steven Dale Cutkosky

MSC:
12D15
13J30
14P05

Keywords:
Orderings
Real places
Formal power series

ABSTRACT

We consider the space of orderings of the field $R((x,y))$ and the space of orderings of the field $R((x))(y)$, where R is a real closed field. We examine the structure of these objects and their relationship to each other. We define a cyclic 2-structure to be a pair (S,Φ) where S is a cyclically ordered set and Φ is an equivalence relation on S such that each equivalence class has exactly two elements. We show that each of these spaces of orderings is described by a cyclic 2-structure, in a natural way. We also show that if the real closed field R is archimedean then the space of $\mathbb{R}$-places of these fields is describable in terms of the cyclic 2-structure.

1. Introduction

For a formally real field K, $\operatorname{Sper} K$ denotes the set of orderings of K, M_K denotes the set of $\mathbb{R}$-places of K, and $\lambda : \operatorname{Sper} K \to M_K$ denotes the natural map. See [3,15,16] or [20] for a more precise description of these objects and for basic terminology and basic results. $\dot{K}$ denotes the multiplicative group $K\backslash\{0\}$. $\operatorname{Sper} K$ and M_K are topological spaces. $\operatorname{Sper} K$ is a Boolean space. The Harrison sets

$$H_K(f) := \{P \in \operatorname{Sper} K \mid f \in P\}, \quad f \in \dot{K},$$

* Corresponding author.
E-mail addresses: salma.kuhlmann@uni-konstanz.de (S. Kuhlmann), marshall@math.usask.ca (M. Marshall), kosiak@math.us.edu.pl (K. Osiak).

0021-8693/$ – see front matter
doi:10.1016/j.jalgebra.2011.02.026

form a subbasis for the topology on $\operatorname{Sper} K$. M_K is compact and Hausdorff. λ is continuous and surjective. The topology on M_K is the quotient topology.

For what we do here, knowledge of abstract spaces of orderings [2,16] is optional. All we need is the definition of the space of orderings of a formally real field. For $f \in \dot{K}$, define $\overline{f} : \operatorname{Sper} K \to \{-1, 1\}$ by

$$\overline{f}(P) := \begin{cases} 1 & \text{if } f \in P, \\ -1 & \text{if } f \in -P. \end{cases}$$

The topology on $\operatorname{Sper} K$ is the weakest topology making the functions $\overline{f}$ continuous, giving $\{-1, 1\}$ the discrete topology. The *space of orderings* of K is the pair $(\operatorname{Sper} K, G_K)$, where G_K is the group of all functions $\overline{f}$, $f \in \dot{K}$.

Orderings and real places arise most naturally in the context of real algebraic geometry [2,4,5,13, 17,20]. Let R be a real closed field, e.g., take $R = \mathbb{R}$. The formal power series ring $R[[x_1, \ldots, x_d]]$ also arises naturally in this context, as the completion of the coordinate ring of a d-dimensional algebraic variety over R at a non-singular point. $R((x_1, \ldots, x_d))$ denotes the field of fractions of the integral domain $R[[x_1, \ldots, x_d]]$.

We restrict our attention here to the case $d = 2$. Orderings on $\mathbb{R}((x, y))$ and on $\mathbb{R}((x, y))_{\mathrm{an}}$, the field of fractions of the ring $\mathbb{R}[[x, y]]_{\mathrm{an}}$ of convergent power series, are considered already in [1]. More recently, in [8], orderings on $\mathbb{R}((x, y))$ are exploited to prove a representation result for polynomials non-negative on a compact basic semialgebraic subset of $\mathbb{R}^2$, extending an earlier such result in [22].

Our main results are Theorems 5.3 and 6.5. The study of orderings and $\mathbb{R}$-places on $R((x, y))$ reduces by an application of the Weierstrass Preparation Theorem, see Theorem 2.1, to the study of orderings and $\mathbb{R}$-places on $R((x))(y)$. It is a consequence of this that the structure of the space of orderings and of the space of $\mathbb{R}$-places of these two fields are closely interrelated. We introduce the idea of a cyclic 2-structure in Section 5 and show, in Theorem 5.3, how each of these spaces of orderings is described by a cyclic 2-structure, in a natural way. In Section 6, which is the most technically demanding section in the paper, we apply ideas from [14] to understand the fibers of the map λ in this situation. We explain, in Theorem 6.5, how the space of $\mathbb{R}$-places is describable in terms of the cyclic 2-structure if R is archimedean. This is an interesting result, more especially so in view of the well-known fact that the space of $\mathbb{R}$-places is typically *not* describable in terms of the space of orderings. We give an example, see Example 6.6, showing how Theorem 6.5 fails if R is not archimedean.

Denote by $R((x, y))_{\mathrm{alg}}$ the field of fractions of the ring $R[[x, y]]_{\mathrm{alg}}$ of algebraic power series [5, Chapter 8]. We do not consider $\mathbb{R}((x, y))_{\mathrm{an}}$ or $R((x, y))_{\mathrm{alg}}$ explicitly in what we do here. But it still needs to be mentioned that everything we do here for $R((x, y))$ carries over with suitable modifications to these fields.

In [11] and [12] it is asked if the pp conjecture holds for the space of orderings of $R((x, y))$. We do not consider this question, although the results we do obtain might provide the basis for an eventual answer to this question.

2. Preparation Theorem and factorization

Throughout the paper R denotes a real closed field. The results in Section 2 are well known and are valid for any field R.

A monic polynomial $f \in R[[x]][y]$ of the form

$$f = y^n + \sum_{i=0}^{n-1} a_i(x) y^i, \quad a_i(x) \in R[[x]],\ x \mid a_i(x),\ 0 \leqslant i < n,\ n \geqslant 0$$

will be called *distinguished*.

Theorem 2.1 *(Preparation Theorem). Every non-zero element $f \in R[[x, y]]$ has a unique decomposition*

$$f = ux^k f^\star,$$

where u is a unit in $R[[x, y]]$, $k \geqslant 0$ and $f^\star$ is a distinguished polynomial in $R[[x]][y]$.

See [23, Corollary 1, p. 145] for the proof. See [23, Corollary 1, p. 131] for a description of the units.

Remark 2.2. The field $R((x))$ is a complete discrete valued field with residue field R. Let $R((x))^{\mathrm{ac}}$ denote the algebraic closure of $R((x))$ and let v denote the unique extension of the valuation to $R((x))^{\mathrm{ac}}$.

(1) Let $f \in R[[x]][y]$ be distinguished, $f = y^n + \sum_{i=0}^{n-1} a_i(x)y^i$. If $r \in R((x))^{\mathrm{ac}}$ and $v(r) \leqslant 0$ then $v(r^n) < v(a_i y^i)$, $i = 1, \dots, n-1$, so $v(f(r)) = v(y^n) \leqslant 0$. In particular, all roots of f have positive value.
(2) Conversely, if $f \in R((x))[y]$ is monic and all the roots of f have positive value then f is distinguished (because the coefficients $a_1, \dots, a_{n-1}$ of f are elementary symmetric functions of the roots, so they also have positive value).
(3) In particular, if $f \in R((x))[y]$ is monic and irreducible and one root of f has positive value then all roots of f have positive value (because the various roots are conjugate to each other, so they have the same value) so f is distinguished.

Lemma 2.3. *If $f \in R[[x]][y]$ is distinguished, then the following conditions are equivalent*:

(1) *f is irreducible in $R[[x, y]]$,*
(2) *f is irreducible in $R[[x]][y]$,*
(3) *f is irreducible in $R((x))[y]$.*

Proof. Since $R[[x]]$ is a UFD and f has content 1 (because it is monic), (2) $\Leftrightarrow$ (3) is clear. (1) $\Rightarrow$ (2): Suppose f is irreducible in $R[[x, y]]$ and $f = gh$, $g, h \in R[[x]][y]$. Scaling by a unit of $R[[x]]$ we may assume g and h are monic so, by Remark 2.2, parts (1) and (2), g and h are distinguished. One of g, h is a unit in $R[[x, y]]$, say g is a unit in $R[[x, y]]$. Since g is also distinguished, this forces $g = 1$, i.e., g is already a unit in $R[[x]][y]$. (2) $\Rightarrow$ (1): By [23, Corollary 2, p. 146], the ring homomorphism $R[[x]][y] \to R[[x, y]]/(f)$ induced by the inclusion $R[[x]][y] \subseteq R[[x, y]]$ is surjective and has kernel equal to the principal ideal in $R[[x]][y]$ generated by f (which, by abuse of notation, we also denote by (f)), so $R[[x, y]]/(f) \cong R[[x]][y]/(f)$. We know that $R[[x]][y]$ is a UFD. If f is irreducible in $R[[x]][y]$ then the principal ideal in $R[[x]][y]$ generated by f is prime, so the principal ideal in $R[[x, y]]$ generated by f is also prime. This implies that f is irreducible in $R[[x, y]]$. $\square$

The ring $R[[x, y]]$ is a UFD [23, Theorem 6, p. 148]. This can be deduced from the fact that $R[[x]][y]$ is a UFD, by combining Theorem 2.1 and Lemma 2.3. Each non-zero $f \in R[[x, y]]$ factors uniquely as

$$f = ux^k f_1 \cdots f_m$$

where u is a unit of $R[[x, y]]$, $k \geqslant 0$, $m \geqslant 0$ and each $f_j \in R[[x]][y]$ is distinguished and irreducible.

We record the following consequence of the proof of Lemma 2.3. See also [23, Corollary, p. 149].

Corollary 2.4. *If $f \in R[[x]][y]$ is distinguished and irreducible, then the field of fractions of $R[[x, y]]/(f)$ is canonically isomorphic to $R((x))[y]/(f)$.*

3. The conjugation map

The field $R((x))$ has two orderings, one making $x>0$, and one making $x<0$. Denote the associated real closures by R_1 and R_2, respectively. Any finite extension L of $R((x))$ is a complete discrete valued field with residue field R or C, where $C := R(\sqrt{-1})$. If the residue field is R then L has two orderings, by the Baer–Krull Theorem [16, Section 1.3], [17, Section 1.5]. If the residue field is C then $\sqrt{-1} \in L$ and L has no orderings. Suppose now that $L = R((x))[y]/(f)$, where $f \in R((x))[y]$ is irreducible. Suppose L is formally real, i.e., the prime ideal (f) is real. Orderings of L correspond to roots of f in $R_1 \dot\cup R_2$ (disjoint union). Either there are two roots of f in R_1 and none in R_2 or two in R_2 and none in R_1 or one in R_1 and one in R_2.

Putting it another way, if $r \in R_1 \dot\cup R_2$ and f denotes the minimal polynomial of r over $R((x))$, then f has another root $r' \in R_1 \dot\cup R_2$. In this way we have a well-defined map $r \mapsto r'$ from $R_1 \dot\cup R_2$ onto itself, which we call the *conjugation map*.

By Puiseux's Theorem, each $r \in R_1$ (resp., $r \in R_2$) is expressible as

$$r = \sum_{i=k}^{\infty} a_i x^{i/d} \quad \left(\text{resp., } r = \sum_{i=k}^{\infty} a_i (-x)^{i/d}\right),$$

$a_i \in R$, $d :=$ the degree of the minimal polynomial of r over $R((x))$. The integer d is also described as the least common denominator of the fractions i/d with $a_i \neq 0$.

By Kummer Theory, for $r = \sum a_i x^{i/d}$, as above, the conjugates of r over $R((x))$ (or equivalently, over $C((x))$) have the form $\sum a_i \omega^i x^{i/d}$ where ω is a d-th root of 1. If d is even, -1 is a d-th root of 1, and $r' = \sum a_i (-1)^i x^{i/d}$. If d is odd then $\mu := -\frac{(-x)^{1/d}}{x^{1/d}}$ is a d-th root of 1, and $r' = \sum a_i \mu^i x^{i/d} = \sum a_i (-1)^i (-x)^{i/d}$. Similar formulas hold for $r = \sum a_i (-x)^{i/d}$.

In summary, the map $r \mapsto r'$ from $R_1 \dot\cup R_2$ to $R_1 \dot\cup R_2$ is given by

$$\sum a_i x^{i/d} \mapsto \sum a_i (-1)^i x^{i/d}, \qquad \sum a_i (-x)^{i/d} \mapsto \sum a_i (-1)^i (-x)^{i/d}$$

if d is even and

$$\sum a_i x^{i/d} \mapsto \sum a_i (-1)^i (-x)^{i/d}, \qquad \sum a_i (-x)^{i/d} \mapsto \sum a_i (-1)^i x^{i/d}$$

if d is odd. If $d=1$ then $r \in R((x))$ so there is one copy of r in R_1 and one in R_2 and, in this case, the map $r \mapsto r'$ just interchanges these two copies.

Remark 3.1. If $R = \mathbb{R}$, the irreducible polynomial $f \in R((x))[y]$ is distinguished and the coefficients of f are analytic functions of x in a neighborhood of 0 then $y = r$ and $y = r'$ (where r, r' are the real conjugate roots of f) are precisely the real half-branches of the plane curve $f(x, y) = 0$ at $(0, 0)$. The same is true for $R \neq \mathbb{R}$, if the irreducible polynomial f is distinguished and the coefficients of f are Nash functions of x in a neighborhood of 0.

Theorem 3.2 *(Continuity of conjugation). For each $r \in R_1 \dot\cup R_2$ and each neighborhood U of $\{r, r'\}$ in $R_1 \dot\cup R_2$, there is a neighborhood V of $\{r, r'\}$ in $R_1 \dot\cup R_2$ contained in U and invariant under conjugation.*

Here, the topology on $R_1 \dot\cup R_2$ is the disjoint union topology, giving each R_i the order topology.

Proof of Theorem 3.2. Since r belongs to R_1 or R_2 and, similarly, r' belongs to R_1 or R_2, there are four cases to consider. We consider the case $r \in R_1$, $r' \in R_1$. The other cases are similar. Thus $r = \sum a_i x^{i/d}$, $r' = \sum a_i (-1)^i x^{i/d}$. Choose $V = V_1 \cup V_2$ where $V_1 := \{s \in R_1 \mid v(s - r) > \gamma\}$ and $V_2 := \{s \in R_1 \mid v(s - r') > \gamma\}$, γ large enough so that $V \subseteq U$ and d is the least common denominator of the fractions $\{i/d \mid i/d < \gamma,\ a_i \neq 0\}$. The point is that if $s \in V$ then s coincides with either r or r' up to terms of value $\geqslant \gamma$ and the degree of s is some multiple of d. If the degree of s is an even multiple of d then s' is in the same part of V as s. If the degree of s is an odd multiple of d then s' is in the other part of V. □

Remark 3.3. Consider the intervals V_i^-, V_i^+, $i = 1, 2$ defined by $V_1^- = \{s \in V_1 \mid s < r\}$, $V_1^+ = \{s \in V_1 \mid s > r\}$, $V_2^- = \{s \in V_2 \mid s < r'\}$, $V_2^+ = \{s \in V_2 \mid s > r'\}$. For each pair V_i^ϵ, V_j^δ, $i, j \in \{1, 2\}$, $\epsilon, \delta \in \{+, -\}$, there are elements of V_i^ϵ which are mapped to V_j^δ by conjugation.

4. Orderings

Let $(S, <)$ be an ordered set. A *cut* of $(S, <)$ is a pair (A, B) where A, B are subsets of S, $A \cup B = S$, and $A < B$. A cut is said to be *proper* if A and B are both non-empty. The two *principal cuts* determined by an element $r \in S$ are

$$r_- := \big(\{a \mid a < r\}, \{b \mid b \geqslant r\}\big) \quad \text{and} \quad r_+ := \big(\{a \mid a \leqslant r\}, \{b \mid b > r\}\big).$$

The set of cuts of an ordered set $S = (S, <)$ will be denoted by $C(S)$. The following result appears to be well known.

Lemma 4.1. *For any ordered set S, the set of cuts of S equipped with its natural order topology is a Boolean space.*

Proof. Define $\Psi : C(S) \to \{0, 1\}^S$ by

$$\Psi(A, B)(r) = \begin{cases} 0 & \text{if } r \in A, \\ 1 & \text{if } r \in B. \end{cases}$$

One checks that Ψ is injective and that the topology on $C(S)$ is induced by Ψ and the product topology on $\{0, 1\}^S$, giving $\{0, 1\}$ the discrete topology. It follows that $C(S)$ is totally disconnected. In view of Tychonoff's Theorem, to show $C(S)$ is compact it suffices to show the image of $C(S)$ under Ψ is closed in $\{0, 1\}^S$. This is straightforward to check. □

Remark 4.2. For a formally real field K, the set $\operatorname{Sper} K(y)$ is naturally identified with the disjoint union of the sets $\operatorname{Sper} R(y)$, where R runs through the set of real closures of K [9, Lemma 8]. The natural bijection $\dot{\bigcup}_R \operatorname{Sper} R(y) \to \operatorname{Sper} K(y)$ is continuous, where $\dot{\bigcup}_R \operatorname{Sper} R(y)$ is given the topology of the disjoint union. If $\operatorname{Sper} K$ is finite then the disjoint union is compact, and the bijection is a homeomorphism. The orderings of $R(y)$ are naturally identified with the cuts of R [9,10]. The topology on $C(R)$ induced by the Harrison topology on $\operatorname{Sper} R(y)$ coincides with the order topology on $C(R)$.

Let R_1, R_2 be the two real closures of $R((x))$ as defined in the previous section. Consider the topological space of orderings of the field $R((x))(y)$. By Remark 4.2 we have

$$\operatorname{Sper} R((x))(y) = \operatorname{Sper} R_1(y) \,\dot{\cup}\, \operatorname{Sper} R_2(y) = C(R_1) \,\dot{\cup}\, C(R_2).$$

Set $I_j := \{r \in R_j \mid v(r) > 0\}$, $j = 1, 2$. Here, v denotes the extension to R_j of the standard discrete valuation on $R((x))$, i.e., I_j is the set of elements of R_j which are infinitely small relative to elements of R.

We will prove that $\operatorname{Sper} R((x, y))$ is identified with $C(I_1) \dot{\cup} C(I_2)$. We begin by proving some preliminary results. Viewing $R((x))(y)$ as a subfield of $R((x, y))$, we have the natural continuous restriction map $\rho : \operatorname{Sper} R((x, y)) \to \operatorname{Sper} R((x))(y)$.

Lemma 4.3. *The map ρ is injective.*

Proof. Suppose that P_1, P_2 are two different orderings of $R((x,y))$. There exists $f \in R[[x,y]]$ which separates P_1 and P_2. By the Preparation Theorem $f = ux^k f^\star$, where $f^\star$ is a distinguished polynomial of $R[[x]][y]$, $k \geqslant 0$, and u is a unit of $R[[x,y]]$. u has the form $u = a + w$, $a \in R$, $a \neq 0$, w an element of the maximal ideal of $R[[x,y]]$. If $a > 0$ then u is a square, and conversely [17, Proposition 1.6.2]. It follows that u is $\pm$ a square so the sign of u is the same at P_1 and P_2. Consequently, the element $x^k f^\star \in R[[x]][y]$ is also a separating element for P_1 and P_2. $\square$

A unit of $R[[x,y]]$ having the form $u = a + w$, $a \in R$, $a > 0$, w an element of the maximal ideal of $R[[x,y]]$, will be referred to as a *positive unit* of $R[[x,y]]$.

Lemma 4.4. *The image of* Sper $R((x,y))$ *under ρ is a subset of $C(I_1) \dot\cup C(I_2)$.*

Proof. Let P be an ordering of $R((x,y))$. The restriction of P to $R((x))(y)$ extends to $R_j(y)$ for $j = 1$ or 2. Denote this extension by Q. Fix a positive element $r \in R_j$, $v(r) \leqslant 0$. r is bounded below by a positive element a of R. (If $j = 1$, resp., $j = 2$, write $r = bx^{k/d} +$ terms of higher value, resp., $r = b(-x)^{k/d} +$ terms of higher value, where $b \in R$, $b \neq 0$. Take $a = b/2$.) $a \pm y$ is a unit and a square in $R[[x,y]]$ so $a \pm y \in P$. It follows that $r \pm y = (r - a) + (a \pm y) \in Q$. Since this is valid for any positive $r \in R_j$ with $v(r) \leqslant 0$, it follows that the cut of R_j determined by Q is actually a cut of I_j. $\square$

Theorem 4.5. *The map $\rho :$ Sper $R((x,y)) \to C(I_1) \dot\cup C(I_2)$ is a homeomorphism.*

Proof. In view of Lemmas 4.3 and 4.4 it remains to show that each element of $C(I_1) \dot\cup C(I_2)$ is in the image of ρ. We begin by considering the case of a principal cut in I_1 determined by $r \in I_1$. The general case will follow from this by a compactness argument. Let f be the minimal polynomial of r over $R((x))$. By Remark 2.2(3), f is distinguished. By Lemma 2.3, f is irreducible in $R[[x,y]]$. Since $R[[x,y]]$ is UFD, the localization $R[[x,y]]_{(f)}$ is a discrete valuation ring of $R((x,y))$ with residue field equal to the field of fractions of $R[[x,y]]/(f)$ which, by Corollary 2.4, is canonically identified with $R((x))[y]/(f)$. The latter field is a complete discrete valued field with exactly 2 orderings. The ordering we are interested in is the ordering, call it P, on $R((x))[y]/(f)$ induced by the embedding of $R((x))[y]/(f)$ into R_1 defined by $y + (f) \mapsto r$. By the Baer–Krull Theorem, there are exactly 2 orderings of $R((x,y))$ compatible with the discrete valuation ring $R[[x,y]]_{(f)}$ and pushing down to the ordering P. The two orderings of $R((x))(y)$ obtained from these two orderings by restriction are precisely the two orderings of $R((x))(y)$ compatible with the discrete valuation ring $R((x))[y]_{(f)}$ and pushing down to the ordering P on the residue field $R((x))[y]/(f)$. These, in turn, are precisely the two orderings coming from the two principal cuts of R_1 corresponding to r.

Let $i_1 :$ Sper $R_1(y) \hookrightarrow$ Sper $R((x))(y)$ be the canonical restriction. For any non-principal proper cut (A,B) of I_1 consider the family of sets

$$H(r_1,r_2) = \rho^{-1}\big(i_1\big(H_{R_1(y)}(y - r_1) \cap H_{R_1(y)}(r_2 - y)\big)\big),$$

where $H_{R_1(y)}(y - r_1)$ and $H_{R_1(y)}(r_2 - y)$ are Harrison subbasis sets of the topological space Sper $R_1(y)$, $r_1 \in A$, $r_2 \in B$. Since the maps ρ and i_1 are continuous, the sets $H(r_1,r_2)$ are closed, and they are non-empty because $H(r_1,r_2)$ contains the inverse image of the orderings of $R((x))(y)$ determined by the principal cuts associated to r, for every $r_1 < r < r_2$. Note that if $r_1, s_1 \in A$ and $r_2, s_2 \in B$ then $H(r_1,r_2) \cap H(s_1,s_2) = H(\max\{r_1,s_1\}, \min\{r_2,s_2\})$. Thus the family is closed under finite intersections. By compactness of the space of orderings this family has a non-empty intersection.

For improper cuts, consider the families:

$$H(r_1) = \rho^{-1}\big(i_1\big(H_{R_1(y)}(y - r_1)\big)\big), \quad r_1 \in I_1$$

and

$$H(r_2) = \rho^{-1}\big(i_1\big(H_{R_1(y)}(r_2 - y)\big)\big), \quad r_2 \in I_1.$$

Each of these families is a nested family of non-empty closed sets. By compactness, the intersection of each of these families is non-empty.

This shows that the image of ρ contains $C(I_1)$. A similar argument shows that the image of ρ contains $C(I_2)$. $\square$

Here is a less cluttered description of the image of ρ:

Corollary 4.6. *The image of* $\operatorname{Sper} R((x, y))$ *under* ρ *is equal to the set of orderings* P *of* $R((x))(y)$ *satisfying* $a \pm y \in P$ *for all positive* $a \in R$.

Proof. Suppose P is an ordering of $R((x))(y)$ satisfying $a \pm y \in P$ for all positive $a \in R$. P extends to an ordering Q of $R_j(y)$ for $j = 1$ or 2. The argument in the proof of Lemma 4.4 shows that the cut of R_j determined by Q is actually a cut of I_j. Theorem 4.5 then implies P is in the image of ρ. The other inclusion is immediate from the fact that for any positive $a \in R$, $a \pm y$ is a positive unit in $R[[x, y]]$, so it is a square. $\square$

Remark 4.7. Using the Preparation Theorem together with the fact that every unit of $R[[x, y]]$ is $\pm$ a square, we see that the homomorphism $G_{R((x))(y)} \to G_{R((x,y))}$ induced by the inclusion $R((x))(y) \subseteq R((x, y))$ is surjective. Combining this with Corollary 4.6, we see that $(\operatorname{Sper} R((x, y)), G_{R((x,y))})$ is identified via ρ with the subspace $(Y, G_{R((x))(y)}|_Y)$ of $(\operatorname{Sper} R((x))(y), G_{R((x))(y)})$, where

$$Y := \bigcap_{a \in R,\, a>0} \big(H_{R((x))(y)}(a + y) \cap H_{R((x))(y)}(a - y)\big).$$

See [16, pp. 32–33] for basic material on subspaces.

5. Cyclic 2-structures

By a *cyclically ordered set* we mean a set S equipped with a ternary relation such that:

(1) $\forall a, b, c \in S$, $a < b < c \Rightarrow a \neq b \neq c \neq a$.
(2) $\forall a, b, c \in S$, $a < b < c \Rightarrow b < c < a$.
(3) $\forall c \in S$, the set $S \backslash \{c\}$ is totally ordered via $a < b$ iff $a < b < c$.[1]

For a cyclically ordered set S and $a, b \in S$, $a \neq b$, the *interval* (a, b) in S is defined to be the totally ordered set $\{x \in S \mid a < x < b\}$. Cuts of S are defined to be cuts of intervals in S identified in the obvious way. The set of all cuts of a cyclically ordered set S, denoted $C(S)$, is itself a cyclically ordered set. It is a Boolean space which is identified naturally with the Boolean space consisting of all cuts of the totally ordered set $S \backslash \{c\}$ for any $c \in S$; see Lemma 4.1.

By a *cyclic* 2*-structure* we mean a pair (S, Φ) consisting of a cyclically ordered set S together with an equivalence relation Φ on S such that each equivalence class has exactly two elements. A priori no connection between the equivalence relation and the ordering is assumed. For $r \in S$, denote by r' the other element of the equivalence class of r. We refer to r' as the *conjugate* of r. The mapping from S to S defined by $r \mapsto r'$ will be called *the conjugation map*. It is idempotent with no fixed points. Each equivalence class $\{r, r'\}$ determines two arcs $(r, r') = \{s \in S \mid r < s < r'\}$ and $(r', r) = \{s \in S \mid r' < s < r\}$

[1] The idea of a cyclically ordered set is obviously not new. See [19] and [21].

and two continuous functions $f_1, f_2 : C(S) \to \{-1, 1\}$ (called the *atoms* associated to equivalence class $\{r, r'\}$) defined by

$$f_1(x) := \begin{cases} 1 & \text{if } x \text{ is a cut of } (r, r'), \\ -1 & \text{if } x \text{ is a cut of } (r', r) \end{cases}$$

and $f_2 := -f_1$. Note: The principal cuts r_+ and r'_- are to be viewed as cuts of (r, r'). Similarly, the principal cuts r_- and r'_+ are to be viewed as cuts of (r', r). A cyclic 2-structure (S, Φ) will be called *separating* if the atoms corresponding to the equivalence classes separate points in $C(S)$, i.e., if

$$\forall x \neq y \in C(S),\ \exists r \in S \text{ such that } x \text{ is a cut of } \big(r, r'\big) \text{ and } y \text{ is a cut of } \big(r', r\big).$$

We denote by $G_{(S,\Phi)}$ the group of functions $f : C(S) \to \{-1, 1\}$ generated by the constant functions $1, -1$ and the various atoms determined from the various equivalence classes of S.

Lemma 5.1. *For a cyclic* 2*-structure* (S, Φ)*, the following are equivalent*:

(1) (S, Φ) *is separating.*
(2) *The topology on* $C(S)$ *is the weakest such that the atoms corresponding to the equivalence classes are continuous.*
(3) $G_{(S,\Phi)}$ *separates points of* $C(S)$.
(4) *The topology on* $C(S)$ *is the weakest such that the elements of* $G_{(S,\Phi)}$ *are continuous.*

Proof. For $f \in G_{(S,\Phi)}$, denote by $H(f)$ the clopen set $H(f) := \{x \in C(S) \mid f(x) = 1\}$. (1) $\Rightarrow$ (2): Let $x \in C(S)$ and let U be an open set in $C(S)$ containing x. By (1) for each $y \in C(S) \backslash U$ there is some atom f such that $x \in H(f)$, $y \in H(-f)$. By compactness, there exist finitely many atoms $f_1, \dots, f_s$ such that $x \in \bigcap_{i=1}^{s} H(f_1) \subseteq U$. (2) $\Rightarrow$ (1) is a consequence of the fact that $C(S)$ is Hausdorff. (1) $\Leftrightarrow$ (3) is immediate from the definition of $G_{(S,\Phi)}$. The proof of (3) $\Leftrightarrow$ (4) is similar to the proof of (1) $\Leftrightarrow$ (2). □

The space of orderings $(\operatorname{Sper} K, G_K)$ of a formally real field K is said to be *described by the cyclic* 2*-structure* (S, Φ) if there exists a bijection $p : \operatorname{Sper} K \to C(S)$ such that $G_K = \{f \circ p \mid f \in G_{(S,\Phi)}\}$.

Lemma 5.2. *If the space of orderings of a formally real field* K *is described by a cyclic* 2*-structure* (S, Φ) *then*

(1) (S, Φ) *is separating*;
(2) *the associated bijection* $p : \operatorname{Sper} K \to C(S)$ *is a homeomorphism*;
(3) *the pair* $(C(S), G_{(S,\Phi)})$ *is an abstract space of orderings isomorphic to the space of orderings* $(\operatorname{Sper} K, G_K)$ *via the map* p.

The terminology in (3) is explained in detail in [16, Chapter 2]. The reader who does not know this terminology should just ignore (3).

Proof of Lemma 5.2. Since G_K separates points in $\operatorname{Sper} K$, p is a bijection and $G_K = \{f \circ p \mid f \in G_{(S,\Phi)}\}$, it follows that $G_{(S,\Phi)}$ separates points in $C(S)$, so (S, Φ) is separating and the topology on $C(S)$ is the weakest such that elements of $G_{(S,\Phi)}$ are continuous, by Lemma 5.1. As explained in Section 1, the topology on $\operatorname{Sper} K$ is the weakest such that the elements of G_K are continuous. Assertions (2) and (3) are clear at this point. □

Theorem 5.3. *For any real closed field* R*, the spaces of orderings of the fields* $R((x))(y)$ *and* $R((x, y))$ *are described by cyclic* 2*-structures in a natural way.*

Proof. We first give the proof for $R((x))(y)$. Let R_1, R_2 be the two real closures of $R((x))$ defined as in Section 3. Define S to be $R_1 \dot\cup R_2 \dot\cup \{-\infty, \infty\}$ (disjoint union) where $-\infty$ and ∞ are new symbols, and order S cyclically so that $\infty < R_1 < -\infty < R_2 < \infty$. Here, the ordering on R_1 is taken to be the opposite of the usual one and the ordering on R_2 is taken to be the usual one. $C(S)$ is identified with $C(R_1) \dot\cup C(R_2)$ which, as was explained in Section 4, is identified with $\operatorname{Sper} R((x))(y)$. Set up the equivalence relation on S so that ∞ and $-\infty$ are in the same class (note that $\pm\bar{x}$ are the two associated atoms, see Section 1 for the meaning of the bar notation) and, for $r \in S$, $r \neq \pm\infty$, $r' =$ the conjugate of r described in Section 3 (recall that r and r' have the same minimal polynomial f over $R((x))$, and note that $\pm\overline{f}$ are the two associated atoms). $G_{R((x))(y)}$ is generated by elements of the form $\overline{f}$, where f is either a non-zero element of $R((x))$ or a monic irreducible in $R((x))[y]$. Any non-zero $u \in R((x))$ is, up to a square, either ± 1 or $\pm x$. A monic irreducible $f \in R((x))[y]$ is either real or non-real. If f is real it is the minimal polynomial over $R((x))$ of some unique pair $\{r, r'\}$ as above. If f is non-real then f is a sum of two squares in $R((x))[y]$ (see [17, p. 19]), so $\overline{f}$ does not contribute to $G_{R((x))(y)}$ in this case.

The proof for $R((x, y))$ is similar. We take $S = I_1 \dot\cup I_2 \dot\cup \{-\infty, \infty\}$ (disjoint union), where $I_i \subseteq R_i$ is the set of infinitesimal elements of R_i, $i = 1, 2$, notation as in Section 4. We order S cyclically so that $\infty < I_1 < -\infty < I_2 < \infty$. Here, the ordering on I_1 is taken to be the opposite of the usual one and the ordering on I_2 is taken to be the usual one. $C(S)$ is identified with $C(I_1) \dot\cup C(I_2)$ which, by Theorem 4.5, is identified with $\operatorname{Sper} R((x, y))$. Set up the equivalence relation on S as in the previous paragraph. For any unit u of $R[[x, y]]$, $\bar{u}$ is one of the constant functions ± 1. An irreducible f of $R[[x, y]]$ is (up to a unit) either x or a distinguished irreducible. In the latter case, f is real or non-real. If f is real it is the minimal polynomial over $R((x))$ of some unique pair $\{r, r'\}$ as above. If f is non-real then f is a sum of two squares in $R[[x]][y]$, so $\overline{f}$ does not contribute to $G_{R((x,y))}$. $\square$

Remark 5.4. The cyclic 2-structures (S, Φ) considered in Theorem 5.3 satisfy additional constraints. For example:

(1) For each equivalence class $\{r, r'\}$ and each open set U of S containing $\{r, r'\}$, there exist disjoint open intervals V_1, V_2 in S with $r \in V_1$, $r' \in V_2$, $V_1, V_2 \subseteq U$ such that $V_1 \cup V_2$ is a union of equivalence classes.
(2) For V_1, V_2 as in (1), consider the intervals V_i^-, V_i^+, $i = 1, 2$ defined by $V_1^- = \{s \in V_1 \mid s < r\}$, $V_1^+ = \{s \in V_1 \mid s > r\}$, $V_2^- = \{s \in V_2 \mid s < r'\}$, $V_2^+ = \{s \in V_2 \mid s > r'\}$. For each pair V_i^ϵ, V_j^δ, $i, j \in \{1, 2\}$, $\epsilon, \delta \in \{+, -\}$, there are elements of V_i^ϵ which are mapped to V_j^δ by conjugation.

See Theorem 3.2 and Remark 3.3 for the case $r \neq \pm\infty$. The remaining case where $r = \pm\infty$ is dealt with similarly.

There are also constraints coming from the fact that $(C(S), G_{(S,\Phi)})$ is a space of orderings, by Lemma 5.2(3), so it satisfies axioms AX1, AX2 and AX3 (see [16, pp. 21–22]) or, equivalently, axioms (α), (β) and (γ) (see [16, p. 26]). The constraint coming from AX1 is that (S, Φ) is separating, which we have talked about already, in Lemmas 5.1 and 5.2. (α) coincides with AX1. (β) asserts that $C(S)$ is compact, which is something we know already. We will not discuss here the constraints coming from AX2 and AX3 or from (γ).

6. Orderings and $\mathbb{R}$-places

Let K be a formally real field, $\operatorname{Sper} K$ the topological space of orderings of K, M_K the space of $\mathbb{R}$-places of K, $\lambda : \operatorname{Sper} K \to M_K$ the natural map. Recall that λ is continuous and surjective [15,16,20]. A subset Y of $\operatorname{Sper} K$ is called a *fan* if $Y \neq \emptyset$ and every character χ of the group $\dot{K}/\bigcap\{\dot{P} \mid P \in Y\}$ such that $\chi(-1) = -1$ is a signature of some ordering $P \in Y$. Here, $\dot{P} := P \backslash \{0\}$. A fan $Y \subseteq \operatorname{Sper} K$ is said to be *trivial* if it contains at most 2 orderings. The *stability index* $s(K)$ of K is defined as the maximum $n \in \mathbb{N}$ such that there exists a fan $Y \subseteq \operatorname{Sper} K$ which contains 2^n orderings (or ∞ if no such finite n exists). There are various equivalent definitions of the stability index; see [6] and [7] or [2] or [15] or [16].

Interest in the stability index derives, in no small part, from its application to minimal generation of semialgebraic sets and semianalytic sets. This is explained in detail in [2]. The following result is well known.

Theorem 6.1.

(1) *The stability index of* $R((x))(y)$ *is equal to* 2.
(2) *The stability index of* $R((x,y))$ *is equal to* 2.

Proof. Any finite extension L of $R((x))$ which is formally real has two orderings, so has stability index 1. It follows from this using [6, Satz 4.6] (see also [2, Theorem 2.7, Chapter 6]) that the stability index of $R((x))(y)$ is at most 2. (Note: There is a misprint in the statement of [6, Satz 4.6]; $s(K)$ should be $s(F)$.) There are lots of 4-element fans in $\operatorname{Sper} R((x,y))$, e.g., if $f \in R[[x,y]]$ is an irreducible which is distinguished and real, the orderings of $R((x,y))$ compatible with the DVR $R[[x,y]]_{(f)}$ form a 4-element fan.

Claim: For any fan Y in $\operatorname{Sper} R((x,y))$, the image Y' of Y under the natural embedding $\operatorname{Sper} R((x,y)) \hookrightarrow \operatorname{Sper} R((x))(y)$ is a fan in $\operatorname{Sper} R((x))(y)$. Consider the group homomorphism

$$\iota : R((\dot{x}))(y)/\bigcap\{\dot{P}' \mid P' \in Y'\} \to R((\dot{x},y))/\bigcap\{\dot{P} \mid P \in Y\}$$

induced by the inclusion $R((x))(y) \subseteq R((x,y))$. Exploiting the Preparation Theorem and the fact that each unit of $R[[x,y]]$ is $\pm$ a square, we see that ι is surjective. ι is clearly injective. Using these facts together with the fact that Y is a fan we see that Y' is also a fan. This proves the claim.

Putting all these things together yields $2 \leqslant s(R((x,y))) \leqslant s(R((x))(y)) \leqslant 2$, so $s(R((x,y))) = s(R((x))(y)) = 2$. □

By the Baer–Krull Theorem, for each $\xi \in M_K$, the fiber $\lambda^{-1}(\xi)$ is a fan, and the elements of $\lambda^{-1}(\xi)$ are in one-to-one correspondence with characters of the group $V/2V$, where V denotes the value group of the valuation associated to λ. If the stability index of K is equal to n, then every fiber $\lambda^{-1}(\xi)$ contains at most 2^n elements.

Corollary 6.2. *For* K *equal to* $R((x))(y)$ *or* $R((x,y))$, *the fibers* $\lambda^{-1}(\xi)$ *of the map* $\lambda : \operatorname{Sper} K \to M_K$ *have at most* 4 *elements.*

It follows from Corollary 6.2 that the mapping λ is either 1–1, 2–1, or 4–1. At which points is it 1–1? At which points is it 2–1? At which points is it 4–1? We work now to develop a refined version of Corollary 6.2, see Theorem 6.4 below, which answers these questions.

To understand the map $\lambda : \operatorname{Sper} R((x,y)) \to M_{R((x,y))}$, it suffices to understand the map $\lambda : \operatorname{Sper} R((x))(y) \to M_{R((x))(y)}$. We explain this now.

Lemma 6.3. *For any ordering* P *of* $R((x,y))$, *the value group of the valuation of* $R((x,y))$ *associated to* P *coincides with the value group of the valuation of* $R((x))(y)$ *associated to the restriction of* P *to* $R((x))(y)$.

Proof. Any positive unit of $R[[x,y]]$ has the form $a+w$ where a is a positive element of R and w is an element of the maximal ideal of $R[[x,y]]$. For any $n \in \mathbb{N}$, $\frac{1}{n} \pm \frac{w}{a}$ is a unit and a square in $R[[x,y]]$, so $\frac{1}{n} \pm \frac{w}{a} \in P$, i.e., $v_P(\frac{w}{a}) > 0$, i.e., $v_P(a+w) = v_P(a)$, where v_P denotes the valuation of $R((x,y))$ associated to P. The result follows from this, using Theorem 2.1. □

Consider now the commutative diagram

$$\begin{array}{ccc} \operatorname{Sper} R((x,y)) & \longrightarrow & \operatorname{Sper} R((x))(y) \\ \lambda \downarrow & & \downarrow \lambda \\ M_{R((x,y))} & \longrightarrow & M_{R((x))(y)}, \end{array}$$

the horizontal maps coming from the inclusion $R((x))(y) \subseteq R((x,y))$. By Lemma 4.3 the upper horizontal map in injective. Coupling this with Lemma 6.3 and the Baer–Krull Theorem, we see that the lower horizontal map is also injective and, for each $\xi \in M_{R((x,y))}$, if ξ' denotes the restriction of ξ to $R((x))(y)$, then the image of the set $\lambda^{-1}(\xi)$ under restriction is precisely the set $\lambda^{-1}(\xi')$.

We know that $\operatorname{Sper} R((x))(y) = \operatorname{Sper} R_1(y) \dot\cup \operatorname{Sper} R_2(y)$. It follows that any $\mathbb{R}$-place of $R((x))(y)$ is the restriction of some $\mathbb{R}$-place of the field $R_k(y)$, for $k \in \{1,2\}$.

In [14] the extensions of an ordering of a field F to a purely transcendental extension $F(y)$ of F are classified in terms of certain distinguished embeddings into power series fields, and it is explained how the $\mathbb{R}$-places, value groups and residue fields of the extensions can be read off in a concrete way from these embeddings. Over the course of the next several paragraphs we explain the results in [14] that we need in the special case $F = R((x))$.

The field $F := R((x))$ has exactly two orderings. Fix one of them, and let $\overline{F}$ be the real closure of F at this ordering, so $\overline{F} = R_k$, $k \in \{1,2\}$, and let V and κ be the associated value group and residue field of F. Note that $V = \mathbb{Z} \times V_0$ (lexicographic product) where V_0 is the value group of R, and $\kappa =$ the residue field of R. The value group and residue field of $\overline{F}$ are $\overline{V} = \mathbb{Q} \times V_0$ and $\overline{\kappa} = \kappa$. Let P be a fixed ordering of $\overline{F}(y)$, let $F' := F(y) = R((x))(y)$, and let V' and κ' be the associated value group and residue field of F'. Let ξ be the $\mathbb{R}$-place on F' determined by P. By the Baer–Krull Theorem, there are exactly $[V' : 2V']$ orderings on F' having $\mathbb{R}$-place equal to ξ.

Fix a proper truncation closed embedding $p_0 : R \hookrightarrow \kappa((V_0))$. Such an embedding always exists [14,18]. Consider the embedding $p_k : \overline{F} \hookrightarrow \kappa((\overline{V}))$, defined by $\sum_i a_i x^i \mapsto \sum_{i,j} a_{ij} x^{(i,j)}$ if $k = 1$, $\sum_i a_i (-x)^i \mapsto \sum_{i,j} a_{ij} x^{(i,j)}$ if $k = 2$, where the a_{ij} are defined by $p_0(a_i) = \sum_j a_{ij} x^j$. This is proper truncation closed and satisfies $p_k(F) \subseteq \kappa((V))$. According to [14, Theorem 1.1], P determines a canonical element $\phi \in \overline{\kappa'}((V'))$, and an extension of p_k to an order preserving embedding $p : \overline{F}(y) \hookrightarrow \overline{\kappa'}((\overline{V'}))$ given by $y \mapsto \phi$. The group V' is generated by V and the support of ϕ. The field κ' is the subfield of $\mathbb{R}$ generated by κ and the coefficients of ϕ. There are three cases to consider:

(1) immediate transcendental case;
(2) residue transcendental case;
(3) value transcendental case.

In the terminology of [14, Theorem 1.1], ϕ is *distinguished*, which means it has the form w, $w + ax^\gamma$, or $w \pm x^\gamma$, depending on which of the three cases one is considering. Here $w = \sum w_\delta x^\delta$, an element of $\kappa((\overline{V}))$. In case (1), $\phi = w$, $w \notin p(\overline{F})$ but every proper truncation of w is in $p(\overline{F})$. In case (2), $\phi = w + ax^\gamma$, $\gamma \in \overline{V}$, $a \in \mathbb{R} \backslash \kappa$, $w \in p(\overline{F})$ and $w_\delta = 0$ for all $\delta \geqslant \gamma$. In case (3), $\phi = w \pm x^\gamma$, $\gamma \notin \overline{V}$, $w \in p(\overline{F})$ and $w_\delta = 0$ for all $\delta > \gamma$.

For any character χ of $V'/2V'$, the map $\sum a_\delta x^\delta \mapsto \sum a_\delta (-1)^{\chi(\delta + 2V')} x^\delta$ defines an automorphism t_χ of the field $\overline{\kappa'}((V'))$. The composite embedding $t_\chi \circ p : F(y) \to \overline{\kappa'}((\overline{V'}))$ induces an ordering on $F(y)$. The canonical element of $\overline{\kappa'}((V'))$ determined by this ordering is $t_\chi(\phi)$. The restriction of $t_\chi \circ p$ to F is either p_1 or p_2. (It is p_k iff $\chi((1,0) + 2V') = 0$.) The orderings on $F(y)$ defined by the composite embeddings $t_\chi \circ p$, $\chi \in \chi(V'/2V')$, are distinct and have the same $\mathbb{R}$-place as P. All orderings on $F(y)$ having the same $\mathbb{R}$-place as P are obtained in this way, as χ runs through the character group $\chi(V'/2V')$.

It is a straightforward matter to write down formulas for the characters of the group $V'/2V'$ in each of the three cases, and also to write down formulas for each of the power series $t_\chi(\phi)$, $\chi \in \chi(V'/2V')$. In this way, everything we have done here can be made very explicit.

We now apply [14, Theorem 5.1], bearing in mind that $V = \mathbb{Z} \times V_0$ where V_0 is divisible, and κ is real closed. In case (1) V'/V is countable (but note that V'/V can be finite only in the case when R is non-archimedean) and $\kappa' = \kappa$. In case (2) V'/V is finite and κ' is purely transcendental over κ of transcendence degree 1. Case (2) cannot occur if $\mathbb{R} \subseteq R$. In case (3) $V' = W \oplus \mathbb{Z}\delta$ where $\mathbb{Z}\delta$ is infinite cyclic, $W \supseteq V$, W/V finite, and $\kappa' = \kappa$.

Theorem 6.4. *The index* $[V' : 2V']$ *is either* 1, 2 *or* 4. *In case* (1) $[V' : 2V'] = 1$ *or* 2 *depending on whether or not* V' *is* 2*-divisible. In case* (2) $V' = \frac{1}{d}\mathbb{Z} \times V_0$, $d \geqslant 1$ *and* $[V' : 2V'] = 2$. *In case* (3) $W = \frac{1}{d}\mathbb{Z} \times V_0$, $d \geqslant 1$ *and* $[V' : 2V'] = 4$.

There is an obvious sufficient condition, expressible in terms of the underlying cyclic 2-structure (S, Φ) defined in Theorem 5.3, for two orderings P and Q to have the same associated $\mathbb{R}$-place. In our next theorem we prove that, in the archimedean case, this sufficient condition is also necessary. This is a nice result, but the proof is rather involved, as there are many cases and subcases to consider.

Theorem 6.5. *Let* P *and* Q *be two distinct orderings of* $R((x))(y)$ *or of* $R((x, y))$.

(1) *A sufficient condition for* P *and* Q *to have the same associated* $\mathbb{R}$*-place is that for each pair of intervals* (r_1, s_1) *and* (r_2, s_2) *of the cyclically ordered set* S *with* $r_1 < P < s_1$ *and* $r_2 < Q < s_2$, *there exists* $r \in S$ *such that* $r_1 < r < s_1$ *and* $r_2 < r' < s_2$.
(2) *If the real closed field* R *is archimedean then the sufficient condition described in* (1) *is also necessary.*

Proof. It suffices to give the proof for the field $R((x))(y)$.

(1) This is more or less clear. Suppose $\lambda(P) \neq \lambda(Q)$. Using the continuity of λ plus the fact that the space of $\mathbb{R}$-places is Hausdorff, there exist open sets U_1 and U_2 in $\operatorname{Sper} R((x))(y)$ with $P \in U_1$, $Q \in U_2$ and $\lambda(U_1) \cap \lambda(U_2) = \emptyset$. Replacing U_1 and U_2 by smaller open sets if necessary, we may assume U_i is defined by some interval (r_i, s_i) in S, for $i = 1, 2$. For any $r \in S$, the principal cuts r_-, r_+, r'_-, r'_+ have the same $\mathbb{R}$-place so we must have $\{r_-, r_+, r'_-, r'_+\} \cap U_i = \emptyset$, for $i = 1$ or 2. It follows that there does not exist $r \in S$ such that $r_1 < r < s_1$ and $r_2 < r' < s_2$.

(2) Suppose now that R is archimedean. Thus $\kappa = R$, $V_0 = \{0\}$, $V = \mathbb{Z}$ and $\overline{V} = \mathbb{Q}$. Suppose $\lambda(P) = \lambda(Q)$ and r_i, s_i are given, $i = 1, 2$, such that $r_1 < P < s_1$ and $r_2 < Q < s_2$. As explained above, [16, Theorem 1.1] implies there are three cases to consider.

Immediate transcendental case. Suppose the embedding corresponding to P is given by $x \mapsto x$, $y \mapsto w$, $w = \sum w_\delta x^\delta \in R((\mathbb{Q}))$. The other case, where the embedding corresponding to P is given by $-x \mapsto x$, $y \mapsto w$ is similar and will be omitted. By definition, $w \notin R_1$ but every proper truncation of w belongs to R_1. Since the value group is $\mathbb{Q}$ and since $\frac{1}{d}\mathbb{Z}$ is cofinal in $\mathbb{Q}$ for any integer $d \geqslant 1$, any proper truncation of w has just finitely many terms. Since Q has the same $\mathbb{R}$-place as P and $Q \neq P$ the Baer–Krull Theorem implies $[V' : 2V'] \geqslant 2$, so $[V' : 2V'] = 2$, by Theorem 6.4. We know that V' is generated over $\mathbb{Z}$ by the exponents of the x^δ appearing in w, by [14, Theorem 1.1], and, since $V' \neq 2V'$, there is some highest 2-power, say it is 2^ℓ, dividing the denominators of the exponents of the x^δ appearing in w. Thus w has the form $w = \sum w_{a/b} x^{a/2^\ell b}$, with $a, b \in \mathbb{Z}$, some a odd, all b odd. Computing $t_\chi(w)$ for the non-trivial character χ of $V'/2V'$, we see that $t_\chi(w) = \sum w_{a/b}(-1)^a x^{a/2^\ell b}$. The embedding corresponding to Q is given by $(-1)^{2^\ell} x \mapsto x$, $y \mapsto \sum w_{a/b}(-1)^a x^{a/2^\ell b}$. There are two cases depending on whether 2^ℓ is even (i.e., $\ell \geqslant 1$) or 2^ℓ is odd (i.e., $\ell = 0$). In either case any sufficiently fine proper truncation r of w satisfies $r_1 < r < s_1$ and $r_2 < r' < s_2$.

Residue transcendental case. Suppose the embedding corresponding to P is given by $x \mapsto x$, $y \mapsto w + ax^\gamma$. The other case, where the embedding corresponding to P is given by $-x \mapsto x$, $y \mapsto w + ax^\gamma$ is similar and will be omitted. Here, $\gamma \in \mathbb{Q}$, $a \in \mathbb{R} \backslash R$, $w \in R_1$ and $w_\delta = 0$ for $\delta \geqslant \gamma$. We know by [14, Theorem 1.1] that V' is generated over $\mathbb{Z}$ by γ and the exponents appearing in w. (Note that the series w has just finitely many terms.) $V' = \frac{1}{d}\mathbb{Z}$ for some integer $d \geqslant 1$. $[V' : 2V'] = 2$ and Q is the ordering determined by the embedding $(-1)^d x \mapsto x$, $y \mapsto t_\chi(w + ax^\gamma)$ where χ is the non-trivial character of $V'/2V'$. There are two cases, depending on whether d is even or odd. Pick r

of the form $r = w + a_1 x^\gamma$, $a_1 \in R$. The point is that, in either case, if we choose a_1 sufficiently close to a then $r_1 < r < s_1$ and $r_2 < r' < s_2$.

Value transcendental case. The embedding corresponding to P has the form $\pm x \mapsto x$, $y \mapsto w \pm x^\gamma$, so there are four cases to consider. We consider only the case $x \mapsto x$, $y \mapsto w + x^\gamma$. The other cases are dealt with similarly. Here, $\gamma \notin \mathbb{Q}$, $w \in R_1$ and $w_\delta = 0$ for all $\delta > \gamma$. By [14, Theorem 1.1] V' is generated by $\mathbb{Z}$, γ and the exponents appearing in w, so $V' = \frac{1}{d}\mathbb{Z} \oplus \mathbb{Z}\gamma$ for some integer $d \geqslant 1$ and $[V' : 2V'] = 4$. d is the least common denominator of the exponents of w, and w is expressible in the form $w = \sum w_i x^{i/d}$. The embedding corresponding to Q is given by $x \mapsto x$, $y \mapsto w - x^\gamma$ or $(-1)^d x \mapsto x$, $y \mapsto \sum w_i (-1)^i x^{i/d} + x^\gamma$ or $(-1)^d x \mapsto x$, $y \mapsto \sum w_i (-1)^i x^{i/d} - x^\gamma$.

Fix an integer ℓ and take r of the form $r = w + x^{\alpha/2^\ell \beta}$ where α, β are odd integers $\geqslant 1$. We claim that for an appropriate choice of ℓ and for $\alpha/2^\ell \beta$ is sufficiently close to γ in the order topology, $r \in (r_1, s_1)$ and $r' \in (r_2, s_2)$. The choice of ℓ depends on which case we are considering. Let 2^m be the highest power of 2 dividing d. If Q is given by $x \mapsto x$, $y \mapsto w - x^\gamma$ choose $\ell = m + 1$, so $r' = w - x^{\alpha/2^\ell \beta}$. If Q is given by $(-1)^d x \mapsto x$ and $y \mapsto \sum (-1)^i w_i x^{i/d} + x^\gamma$, take $\ell = m - 1$, so $r' = \sum (-1)^i w_i x^{i/d} + x^{\alpha/2^\ell \beta}$ if d is even, resp., $r' = \sum (-1)^i w_i (-x)^{i/d} + (-x)^{\alpha/2^\ell \beta}$ if d is odd. If Q is given by $(-1)^d x \mapsto x$ and $y \mapsto \sum (-1)^i w_i x^{i/d} - x^\gamma$ take $\ell = m$, so $r' = \sum (-1)^i w_i x^{i/d} - x^{\alpha/2^\ell \beta}$ if d is even, resp., $r' = \sum (-1)^i w_i (-x)^{i/d} - (-x)^{\alpha/2^\ell \beta}$ if d is odd. □

If the real closed field R is not archimedean then the sufficient condition given in part (1) of Theorem 6.5 is not necessary.

Example 6.6. We know $V = \mathbb{Z} \times V_0$ ordered lexicographically. If R is not archimedean then $V_0 \neq \{0\}$. Fix a proper cut (A, B) of V_0 and take $\gamma = (1, \gamma_0)$ where $A < \gamma_0 < B$. Consider the orderings P and Q of $R((x, y))$ corresponding to the embeddings $x \mapsto x$, $y \mapsto x^{1/2} + x^\gamma$ and $x \mapsto x$, $y \mapsto x^{1/2} - x^\gamma$ respectively. Clearly $\lambda(P) = \lambda(Q)$. Any $r \in R_1$ close to P has the form $r = x^{1/2} + ax + \cdots$ for some $a \in R$, $a > 0$. Then r' has the form $r' = x^{1/2} + ax + \cdots$ or $r' = -x^{1/2} + ax + \cdots$. In either case, r' is not close to Q.

References

[1] M.E. Alonso, J.M. Gamboa, J.M. Ruiz, On orderings in real surfaces, J. Pure Appl. Algebra 36 (1985) 1–14.
[2] C. Andradas, L. Bröcker, J.M. Ruiz, Constructible Sets in Real Geometry, Ergeb. Math. Grenzgeb. (3), vol. 33, Springer-Verlag, 1996.
[3] E. Becker, Valuations and real places in the theory of formally real fields, in: Géométrie Algébrique Réelle et Formes Quadratiques, Proceedings of Conference in Rennes, in: Lecture Notes in Math., vol. 959, Springer-Verlag, 1982, pp. 1–40.
[4] E. Becker, D. Gondard, Notes on the space of real places of a formally real field, in: Real Analytic and Algebraic Geometry, Trento Conference, 1992, Walter de Gruyter, 1995, pp. 21–46.
[5] J. Bochnak, M. Coste, M.-F. Roy, Géométrie Algébrique Réelle, Ergeb. Math. Grenzgeb. (3), vol. 12, Springer-Verlag, 1986.
[6] L. Bröcker, Zur Theorie der quadratischen Formen über formal reellen Körpern, Math. Ann. 210 (1974) 233–256.
[7] L. Bröcker, Characterization of fans and hereditarily pythagorean fields, Math. Z. 51 (1976) 149–163.
[8] J. Cimprič, S. Kuhlmann, M. Marshall, Positivity in power series rings, Adv. Geom. 10 (2010) 135–143.
[9] T. Craven, The topological space of orderings of a rational function field, Duke Math. J. 41 (1974) 339–347.
[10] R. Gilmer, Extension of an order to a simple transcendental extension, in: Contemp. Math., vol. 8, 1981, pp. 113–118.
[11] P. Gładki, The pp conjecture in the theory of spaces of orderings, PhD thesis, Univ. of Saskatchewan, 2007, http://library2.usask.ca/theses/.
[12] P. Gładki, M. Marshall, The pp conjecture for the space of orderings of the field $\mathbb{R}(x, y)$, J. Pure Appl. Algebra 212 (2008) 197–203.
[13] M. Knebusch, C. Scheiderer, Einführung in die reelle Algebra, Vieweg Stud. Aufbaukurs Math., vol. 63, 1989.
[14] F.-V. Kuhlmann, S. Kuhlmann, M. Marshall, M. Zekavat, Embedding ordered fields in formal power series fields, J. Pure Appl. Algebra 169 (2002) 71–90.
[15] T.-Y. Lam, Orderings, Valuations and Quadratic Forms, CBMS Reg. Conf. Ser. Math., vol. 52, Amer. Math. Soc., 1983.
[16] M. Marshall, Spaces of Orderings and Abstract Real Spectra, Lecture Notes in Math., vol. 1636, Springer-Verlag, 1996.
[17] M. Marshall, Positive Polynomials and Sums of Squares, Math. Surveys Monogr., vol. 146, Amer. Math. Soc., 2008.
[18] M.H. Mourgues, J.P. Ressayre, Every real closed field has an integer part, J. Symbolic Logic 58 (1993) 641–647.
[19] V. Novák, Cyclically ordered sets, Czechoslovak Math. J. 32 (1982) 460–473.
[20] A. Prestel, C.N. Delzell, Positive Polynomials, Springer Monogr. Math., 2001.
[21] A. Quilliot, Cyclic orders, European J. Combin. 10 (1989) 477–488.
[22] C. Scheiderer, Sums of squares on real algebraic surfaces, Manuscripta Math. 119 (2006) 395–410.
[23] O. Zariski, P. Samuel, Commutative Algebra, vol. 2, Univ. Ser. Higher Math., Van Nostrand, Princeton, NJ, 1960.